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Translated by L.K.

PMM U.S.S.R.,Vol.51,No.2,pp.208-212,1987
Printed in Great Britain

0021-8928/87 $10.00+0.00

THE EFFECT OF THIRD-AND FOURTH-ORDER MOMENTS OF INERTIA ON THE MOTION OF A SOLID*

R.S. SULIKASHVILI

The problem of the effects of higher-order moments of inertia on the motion of a solid, fixed at the centre of mass and having a spherical central ellipsoid of inertia in a central Newtonian field of force is investigated. Uniform bodies of the simplest geometrical shapes (a cube, cone and cylinder) are considered. In view of the difference in the symmetries of these bodies the nature of their motions will be different. The equations of motion of a cone and a cylinder are integrated in terms of ultra-elliptic and hyperelliptic functions respectively. Sets of positions of equilibrium, permanent rotations, and regular precessions are indicated, and their branching and stability are investigated. Unlike the case when only second-order moments of inertia are taken into account, two features are determined here: 1) tow families of inclined positions with respect to equilibrium exist, and 2) for a body in the form of a cone the direct position of relative equilibrium is unstable if the vertex of the cone is situated between an attracting centre and a fixed point, and is stable otherwise, which has no analogue for permanent rotations of a body with a triaxial central ellipsoid of inertia.

1. Suppose $O\xi\eta\zeta$ is a fixed system of coordinates with origin at the centre of mass of a body at a distance R from an attracting centre and an axis ζ directed along a rising local vertical, and $Ox_1x_2x_3$ is a system of coordinates rigidly coupled to the body. The mutual orientation of the ξ, η, ζ and x_1, x_2, x_3 axes is specified by a matrix of direction cosines. We will denote the unit vectors of the ξ, η, ζ axes by $\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}$, and their projections on to the x_1, x_2, x_3 axes by $\alpha_i, \beta_i, \gamma_i$ $(i = 1, 2, 3)$

The coordinates x_1, x_2, x_3 of a point of the body will be written in dimensionless form by relating them to a characteristic linear dimension a of the body (a is the side of the cube or the radius of the base for a cone and a cylinder).

The force function U of the forces of Newtonian traction has the form (μ is the gravitational constant and ρ is the density of the body)

$$U = \iiint \frac{\mu\rho}{\Delta}\, dx_1\, dx_2\, dx_3 = \frac{\mu\rho}{R} \iiint f(\varepsilon)\, dx_1\, dx_2,\, dx_3 \tag{1.1}$$
$$\Delta = R\,[\varepsilon^2(\xi^2 + \eta^2) + (1 + \varepsilon\zeta)^2]^{1/2} = R\,[1 + 2\varepsilon\,(x_1\gamma_1 + x_2\gamma_2 + x_3\gamma_3) + \varepsilon^2\,(x_1^2 + x_2^2 + x_3^2)]^{1/2} \quad (\varepsilon = a/R \leqslant 1)$$
$$f(\varepsilon) = [1 + 2\varepsilon\zeta + \varepsilon^2\,(\xi^2 + \eta^2 + \zeta^2)]^{-1/2}$$

It can be seen that U is independent of α_i and β_i, and hence equilibrium is preserved as the body rotates about the ζ axis.

We will calculate U up to fourth-order terms in ε using the expansion

*Prikl.Matem.Mekhan.,51,2,268-274,1987

$$f(\varepsilon) = 1 + \varepsilon f' + \frac{\varepsilon^2}{2} f'' + \frac{\varepsilon^3}{6} f''' + \frac{\varepsilon^4}{24} f'''' + \ldots$$
$$f' = \zeta, \quad f'' = 3\zeta^2 - r^2, \quad f''' = -15\zeta^3 + 9\zeta r^2$$
$$f'''' = 105\zeta^4 - 90\zeta^2 r^2 + 9r^4, \quad r^2 = \xi^2 + \eta^2 + \zeta^2$$

2. The corresponding expression for the principal term U of the force function (1.1) for a cube, assuming that the coordinate planes of the system of coordinates $Ox_1x_2x_3$ are parallel to the edges of the cube, has the form (m is the mass of the body)

$$U = \varkappa \{\gamma_1^2\gamma_2^2 + (\gamma_1^2 + \gamma_2^2)[1 - (\gamma_1^2 + \gamma_2^2)]\}, \quad \varkappa = 7\mu m\varepsilon^4/96R$$

The equations of equilibrium of the cube

$$\frac{\partial U}{\partial \gamma_1} = -2\varkappa(2\gamma_1^3 + \gamma_1\gamma_2^2 - \gamma_1) = 0, \quad \frac{\partial U}{\partial \gamma_2} = -2\varkappa(2\gamma_2^3 + \gamma_1^2\gamma_2 - \gamma_2) = 0$$

have the following three groups of solutions:

$$\text{a) } \gamma_1^2 = 1, \quad \text{b) } \gamma_1 = 0, \quad \gamma_2^2 = \gamma_3^2 = 1/2, \quad \text{c) } \gamma_1^2 = \gamma_2^2 = \gamma_3^2 = 1/3 \qquad (123)$$

The family of solutions a) contains six positions of equilibrium in which the ζ axis passes through the centre of the face; family b) contains twelve equilibria in which the ζ axis passes through the middle of an edge; family c) contains eight equilibira in which the ζ axis passes through the vertex of the cube.

By calculating the second variation $\delta^2 U$ of the function U, it can be shown that the family of equilibria c) is stable, while a) and b) are unstable, and the degree of instability is equal to 1 and 2 respectively for these.

3. To obtain the force function of a cone and a cylinder we will direct the x_3 axis along the axis of symmetry of these bodies. The height of the cone and the cylinder, determined from the condition for the second-order moments of intertia to be equal, is $2a$ and $\sqrt{3}a$ respectively.

For a cone and a cylinder the principal terms in the expansion of the force function have the following form respectively:

$$U_1 = k_0(3 - 5\gamma_3^2)\gamma_3 \quad (k_0 = \mu m\varepsilon^2/(16R)) \tag{3.1}$$
$$U_2 = \nu_0(6 - 7\gamma_2^3)\gamma_3^2 \quad (\nu_0 = 11\mu m\varepsilon^4/(128R)) \tag{3.2}$$

It is obvious that the positions of equilibrium of the cone and the cylinder are independent of the angles of rotation of the bodies about the ζ and x_3 axes.

The equations of equilibrium of the cone and the cylinder can be written in the form

$$dU_1/d\theta = 3k_0(5\cos^2\theta - 1)\sin\theta = 0 \tag{3.3}$$
$$dU_2/d\theta = 2\nu_0(7\cos^2\theta - 3)\sin 2\theta = 0 \quad (\gamma_3 = \cos\theta) \tag{3.4}$$

and have the following solutions:

$$\text{a) } \cos\theta = -1, \quad \text{b) } \cos\theta = 1/\sqrt{5}, \quad \text{c) } \cos\theta = 1\sqrt{5}, \text{d) } \cos\theta = 1 \tag{3.5}$$
$$\text{a) } \cos\theta = -1, \quad \text{b) } \cos\theta = -\sqrt{3/7}, \quad \text{c) } \cos\theta = 0, \tag{3.6}$$
$$\text{d) } \cos\theta = \sqrt{3/7}, \quad \text{e) } \cos\theta = 1$$

The problem of the stability of the equilibria (3.5) and (3.6) can be solved by investigating the sign of the second derivative with respect to U_1 and U_2. It can be shown that the equilibria of the cone a) and b) are stable, while the equilibria c) and d) are unstable, the equilibria of the cylinder b) and d) are stable, while the equilibria a), c) and e) are unstable.

4. We will investigate the permanent rotations and regular precessions of the cone and the cylinder in Euler variables.

For both cases the kinetic energy

$$T = 1/2\, A \mid \theta^{\cdot 2} + \psi^{\cdot 2} + \varphi^{\cdot 2} + 2\varphi^{\cdot}\psi^{\cdot}\cos\theta) \tag{4.1}$$

and the force functions (3.1) and (3.2) are independent of φ and ψ. Consequently, the equations of motion have the following cyclic integrals

$$\omega_3 = \varphi^{\cdot} + \psi^{\cdot}\cos\theta = c_1, \quad \omega_\zeta = \varphi^{\cdot}\cos\theta + \psi^{\cdot} = c_2 \tag{4.2}$$

where ω_3 and ω_ζ are the projections of the instantaneous angular velocity ω of the x_3 and ζ

axes. The equation for θ reduces to the following form: for a cone

$$\theta^{\cdot\cdot} + \varphi^{\cdot}\psi^{\cdot} \sin\theta = k \sin\theta\,(5\cos^2\theta - 1) \quad (k = 3k_0/A) \tag{4.3}$$

and for a cylinder

$$\theta^{\cdot\cdot} + \varphi^{\cdot}\psi^{\cdot} \sin\theta = \nu \sin\theta \cos\theta\,(7\cos^2\theta - 3)\ (\nu = 4\nu_0/A) \tag{4.4}$$

Eqs.(4.2), (4.3), and (4.2), (4.4) are integrable. Their general solution can be obtained by inversion of the ultraelliptic and hyperelliptic integrals respectively

$$d\gamma_3/dt = \pm [p_5(\gamma_3)]^{1/2}, \quad d\gamma_3/dt = \pm [p_6(\gamma_3)]^{1/2}$$
$$p_5(\gamma_3) = h\,(1-\gamma_3^2) + 2k\gamma_3\,(1-\gamma_3^2)(3-5\gamma_3^2) + 2c_1c_2\gamma_3 - (c_1^2 + c_2^2)$$
$$p_6(\gamma_3) = h\,(1-\gamma_3^2) + 2\nu\gamma_3^2\,(1-\gamma_3^2)(6-7\gamma_3^2) + 2\,c_1c_2\gamma_3 - (c_1^2 + c_2^2)$$

Eqs.(4.2), (4.3) and (4.2), (4.4) have a family of partial solutions which are shown in the table ($\lambda = 1/\sqrt{5}$ for a cone and $\lambda = \sqrt{3/7}$ for a cylinder), and determine the permanent rotations of the cone and the cylinder.

Table 1

№	$\cos\theta$	$\varphi^{\cdot}$	$\psi^{\cdot}$	Notes	№	$\cos\theta$	$\varphi^{\cdot}$	$\psi^{\cdot}$	Notes
1°	0	0	c_2	only for	5°	λ	0	c_2	$c_1 = \lambda c_2$
2°	0	c_1	0	a cylinder	6°	λ	c_1	0	$c_2 = \lambda c_1$
3°	1			$\omega_3 = c_1 = c_2$	7°	$-\lambda$	0	c_2	$c_1 = -\lambda c_2$
4°	-1			$\omega_3 = c_2 = -c_1$	8°	$-\lambda$	c_1	0	$c_2 = -\lambda c_1$

For solutions 5°-8° the axis of the cone is inclined to the ζ axis and the cone rotates with constant angular velocity $\psi^{\cdot} = c_2$ around the ζ axis, or $\varphi^{\cdot} = c_1$ around the x_3 axis.

In addition to these solutions, Eq.(4.2) and (4.3) also have a family of solutions of the form $\varphi^{\cdot} = \varphi_0^{\cdot}$, $\psi^{\cdot} = \psi_0^{\cdot}$, $\theta^{\cdot} = 0$, $\theta = \theta_0$, which exist when the conditions $-k \leqslant \varphi_0^{\cdot}\psi_0^{\cdot} \leqslant 4k$ are satisfied. Regular precessions of the cone correspond to these solutions. If $\varphi^{\cdot} = 0$ or $\psi^{\cdot} = 0$, the regular precessions of the cone become permanent rotations. If instead of $\varphi_0^{\cdot}, \psi_0^{\cdot}$ we take c_1 and c_2 as the parameters, we will have the following equations for determining θ_0:

$$(c_1 - c_2\beta)(c_2 - c_1\beta) + k\,(1-\beta^2)(1-5\beta^2) = 0, \quad \beta = \cos\theta \tag{4.5}$$

It is obvious that for any c_1 and c_2 $(c_1 \neq \pm c_2)$, Eq.(4.5) has at least one real root $\theta = \theta_0$ $(0 < \theta_0 < \pi)$, since the function on the left-hand side of (4.5) changes sign in the interval $[-1, 1]$.

Consider the case of a cylinder. For solutions 1° and 2° the axis of symmetry of the cylinder x_3 is perpendicular to the ζ axis, and the cylinder in this case rotates with an arbitrary angular velocity $\psi_0^{\cdot}$ around the ζ axis, or $\varphi_0^{\cdot}$ around the x_3 axis. In cases 5°-8° the axis of the cylinder is inclined to the ζ axis and the cylinder rotates with arbitrary angular velocity $\psi_0^{\cdot}$ around the ζ axis, or $\varphi_0^{\cdot}$ around the x_3 axis.

In addition to these solutions, Eqs.(4.2) and (4.4) also have a family of solutions of the form $\varphi^{\cdot} = \varphi_0^{\cdot}$, $\psi^{\cdot} = \psi_0^{\cdot}$, $\theta^{\cdot} = 0$, $\theta = \theta_0$ describing regular precessions of the cylinder. The values θ_0 are found from the equation

$$(c_1 - c_2\beta)(c_2 - c_1\beta) + \nu\beta\,(1-\beta^2)(1+\beta)^2(3-7\beta^2) = 0, \quad \beta = \cos\theta \tag{4.6}$$

Like (4.5), Eq.(4.6) for any c_1 and c_2 $(c_1 \neq \pm c_2)$ has at least one real root $\theta = \theta_0$ $(0 < \theta_0 < \pi)$.

If, instead of c_1 and c_2 we take $\varphi_0^{\cdot}$ and $\psi_0^{\cdot}$ as the parameters, we will obtain the following equation for θ_0:

$$7\cos^3\theta_0 - 3\cos\theta_0 - \varphi_0^{\cdot}\psi_0^{\cdot}/\nu = 0 \tag{4.7}$$

When the conditions

$$|\varphi_0^{\cdot}\psi_0^{\cdot}| < 2\nu/\sqrt{7}, \; -\sqrt{7/2} \leqslant \cos\,[\arccos\,(7\varphi_0^{\cdot}\psi_0^{\cdot} + 4l\nu\pi)/6] \leqslant \sqrt{7/2}, \quad l = 0, 1, 2$$

are satisfied, Eq.(4.7) has three different real roots; when $|\varphi_0^{\cdot}\psi_0^{\cdot}| = 2\nu/\sqrt{7}$ there are three real roots $\cos\theta_{01} = 2M^{1/3}$, $\cos\theta_{02} = \cos\theta_{03} = M^{1/3}$, two of which are identical; when $|\varphi_0^{\cdot}\psi_0^{\cdot}| > 2\nu/\sqrt{7}$ and $-1 \leqslant \alpha^+ + \alpha^- \leqslant 1$ there is one real root $\cos\theta_0 = \alpha^+ + \alpha^-$.

Here $\alpha^{\pm} = (M \pm N)^{1/3}$, $M = \varphi_0^{\cdot}\psi_0^{\cdot}/(14\nu)$, $N = \sqrt{M^2 - 1/7}/7$

5. The sets of permanent rotations and regular precessions of the cone and the cylinder

can be represented geometrically in the space of the variables $c_1, c_2, \theta(\gamma_3)$ in the form of surfaces defined by Eqs.(4.5) and (4.6). To analyse these surfaces we will write Eqs.(4.5) and (4.6) in the form

$$(1-\gamma_3)^2 m^2 - (1+\gamma_3)^2 n^2 = 6k\,(1-\gamma_3)^2 (1+\gamma_3)^2 (5\gamma_3^2 - 1) \tag{5.1}$$

$$(1-\gamma_3)^2 m^2 - (1+\gamma_3)^2 n^2 = 2\nu\gamma_3\,(1-\gamma_3)^2 (1+\gamma_3)^2 (7\gamma_3^2 - 3) \tag{5.2}$$

$$c_1 = (m-n)/\sqrt{2}, \quad c_2 = (m+n)/\sqrt{2}$$

The sections of the surfaces (5.1) and (5.2) with the planes $\gamma_3 = \gamma_{30}$ **represent hyperbola,** the principal axes of which are the coordinate axes m and n, if γ_{30} does not reduce the right-hand sides of Eqs.(5.1) and (5.2) to zero; otherwise, we will have a pair of intersecting straight lines $(1-\gamma_{30})\,m = \pm(1+\gamma_{30})\,n$ in the section. **The points of the hyperbolas correspond** to regular precessions, while the points of the straight lines correspond to permanent rotations.

In Figs.1-4 we show sections of the surfaces (5.1) by the planes $n=0$ (Fig.1), $m=0$ (Fig.2), $n = \pm\delta_1 m$ Fig.3), $n = \pm\delta_2 m$ (Fig.4), $(\delta_1 = \delta_2^{-1} = (\sqrt{5}-1)(\sqrt{5}+1)^{-1})$. In all the figures the points on the γ_3 axis for which $\gamma_3 = \pm 1, \pm 1/\sqrt{5}$ correspond to positions of equilibrium of the cone. Its permanent rotations correspond to the straight lines $\gamma_3 = 1$ (Fig.1), $\gamma_3 = -1$ (Fig.2), $\gamma_3 = 1/\sqrt{5}$ (Fig.3), and $\gamma_3 = -1/\sqrt{5}$ (Fig.4).

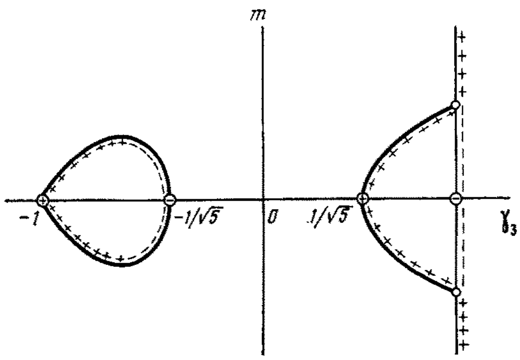

Fig.1

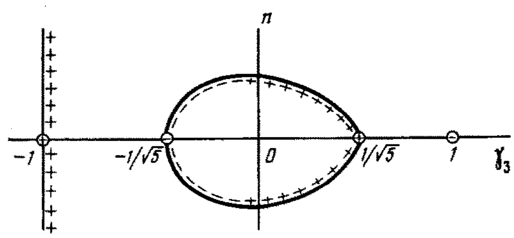

Fig.2

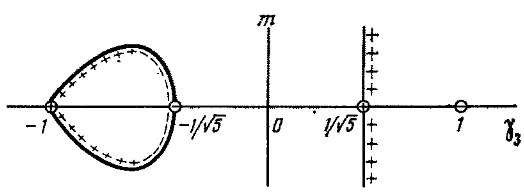

Fig.3

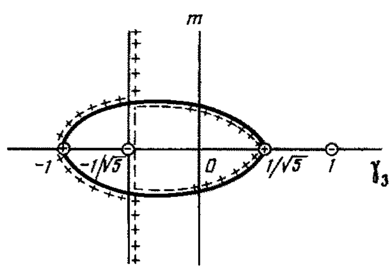

Fig.4

In Figs.5-9 we show sections of the surface (5.2) by the planes $n=0$ (Fig.5), $m=0$ (Fig.6), $n = \pm m$ (Fig.7), $n = \pm\delta_3 m$ (Fig.8), and $n = \pm\delta_4 m$ (Fig.9) $(\delta_3 = \delta_4^{-1} = (\sqrt{7}-\sqrt{3})(\sqrt{7}+\sqrt{3})^{-1})$.

In all Figs.5-9 points on the γ_3 axis for which $\gamma_3 = 0, \pm 1, \pm\sqrt{3/7}$ correspond to positions of equilibrium of the cylinders. The straight lines $\gamma_3 = 1$ (Fig.5), $\gamma_3 = -1$ (Fig.6), $\gamma_3 = 0$ (Fig.7), $\gamma_3 = \sqrt{3/7}$ (Fig.8), and $\gamma_3 = -\sqrt{3/7}$ (Fig.9) correspond to its permanent rotations.

The remaining branches in Figs.1-9 correspond to regular precessions of the cone and the cylinder.

We will investigate the stability of the permanent rotations and regular precessions of the cone and the cylinder. Let us consider Fig.1 to be specific. In it the points $(\gamma_3 = 1, m=0)$ and $(\gamma_3 = -1/\sqrt{5}, m=0)$, as was shown in Sect.3, correspond to unstable positions of equilibrium of the cone, while the points $(\gamma_3 = 1/\sqrt{5}, m = 0)$ and $(\gamma_3 = -1, m = 0)$ correspond to stable equilibria. On the basis of the theory of bifurcations we conclude that the branches shown in Fig.1 by the plus and minus signs correspond to stable and unstable permanent rotations and regular precessions of the cone. A change in stability occurs at the points of bifurcation.

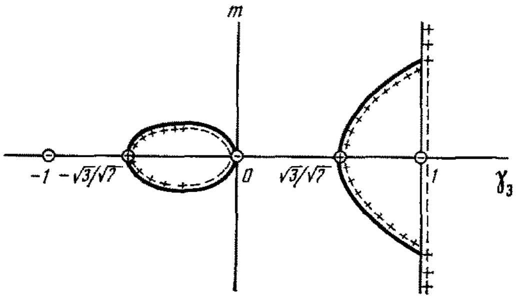

Fig.5

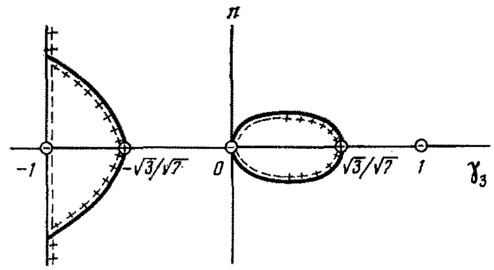

Fig.6

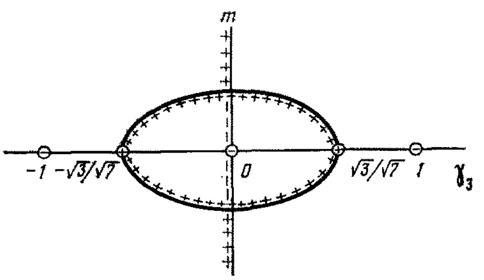

Fig.7

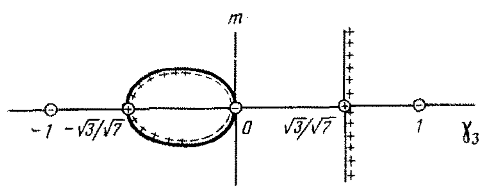

Fig.8

Similar conclusions can be reached regarding the stability of the permanent rotations and regular precessions for other possible cases. The results of the analysis of the stability and instability are shown in Figs.1-9.

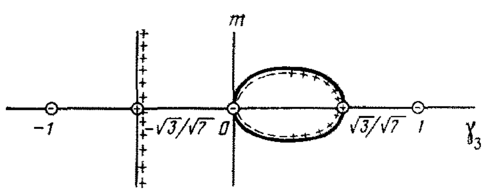

Fig.9

Note that the permanent rotations of the cone and the cylinder corresponding to unstable equilibrium orientations are also unstable for a fairly low angular velocity, and stable for fairly high angular velocity.

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Translated by R.C.G.